García, J.L., Gómez Sánchez, P.L. and Martínez Hernández, J.
Osaka J. Math.
42 (2005), 173–187

LOCALLY FINITELY PRESENTED CATEGORIES AND FUNCTOR RINGS

J.L. GARCÍA, P.L. GÓMEZ SÁNCHEZ and J. MARTÍNEZ HERNÁNDEZ

(Received June 25, 2003)

Abstract

By using the correspondence between locally finitely presented additive categories and rings with enough idempotents, we study several properties of such rings in terms of the associated categories, and conversely. In particular, it is shown that a ring R (with enough idempotents) is right perfect and the categories of finitely presented right and left R-modules are dual to each other if and only if the categories of projective and of injective right R-modules are equivalent.

1. Introduction

An important tool in the study of purity has been the use of Gabriel's functor rings. In 1994, Crawley-Boevey [5] gave a quite general version of this technique, by introducing the concept of a locally finitely presented additive category (briefly, an l.f.p. additive category). First, an object M of an additive category $\mathcal{A}$ is finitely presented if the functor $\mathrm{Hom}_{\mathcal{A}}(M, -)$ preserves direct limits. Then, we shall say that the additive category $\mathcal{A}$ is locally finitely presented in case every directed system of objects and morphisms has a direct limit, the class of finitely presented objects of $\mathcal{A}$ is skeletally small and every object of $\mathcal{A}$ is the direct limit of finitely presented objects. One may define the functor ring R (see [9]) associated to the category $\mathcal{A}$ as the ring

$$R = \bigoplus_{\lambda\in\Lambda} \bigoplus_{\mu\in\Lambda} \mathrm{Hom}(U_\lambda, U_\mu)$$

where $\{U_\lambda\}_{\lambda\in\Lambda}$ is a set of isomorphism classes of finitely presented objects of $\mathcal{A}$. With the natural sum and multiplication, R is a ring with enough idempotents

$$R = \bigoplus_{\Lambda} e_\lambda R = \bigoplus_{\Lambda} Re_\lambda$$

e_λ being the identity on U_λ. The category $\mathcal{A}$ may be embedded as a full subcategory of the category $\mathrm{Mod}(R)$ of all the unitary right R-modules (i.e., modules M_R such that $MR = M$) in such a way that pure exact sequences in $\mathcal{A}$ are those that are carried into exact sequences of $\mathrm{Mod}(R)$ through the embedding; and $\mathcal{A}$ can be thus seen as the category of flat right R-modules. As an example of the use of this technique, it was shown by Simson [18] that a locally finitely presented Grothendieck category $\mathcal{A}$ is pure semisimple (i.e., all pure exact sequences of $\mathcal{A}$ split) if and only if its associated functor ring is right perfect.

It was more or less implicit in this construction that the association of the ring R to the l.f.p. additive category $\mathcal{A}$ is one-to-one. In [7, Theorem 1.1] this was explicitly shown: there exists, up to equivalence, a bijection between the class of all l.f.p. additive categories and the class of all rings with enough idempotents. In this bijection, the category corresponding to a given ring R is the category $Fl(R)$ of all flat right R-modules. It is the very existence of this bijection what justifies the possibility of relating properties of an l.f.p. additive category to properties of its associated functor ring.

The power of this technique is therefore not limited to the study of purity. In view of the above bijection, we may study arbitrary properties of l.f.p. additive categories by means of their associated functor rings, in a systematic way. To show the possibility and interest of this study is our purpose in this paper. We shall illustrate this technique with several examples, characterizing certain classes of locally finitely presented additive categories in terms of their associated functor rings. Equivalently, we may also characterize properties of an arbitrary ring R with enough idempotents in terms of properties of the category $Fl(R)$. Results containing such characterizations are given in Sections 2 and 3. In the last section, we use these same techniques to study a class of rings with enough idempotents that is a natural generalization of the class of unital quasi-Frobenius rings.

In what follows, ring will mean ring with enough idempotents and module will mean unitary module. The category Mod(R) (respectively, Mod(R^{op})) is the category of all unitary right (resp., left) R-modules.

2. Some classes of locally finitely presented additive categories

Given a locally finitely presented additive category $\mathcal{A}$, the problem of when is $\mathcal{A}$ an abelian category has been extensively studied. In this connection, there has also been some interest in studying when the category $\mathcal{A}$ has kernels, cokernels, products, and so on. Let us recall that a morphism $f: X \to Y$ of the l.f.p. additive category $\mathcal{A}$ is said to be a pseudokernel of the morphism $g: Y \to Z$ if $g \circ f = 0$, and whenever $g \circ h = 0$ for some morphism h, then $h = f \circ m$ for some morphism m. Dually, one defines a pseudocokernel. On the other hand, if $\mathcal{C}$ is a class of objects of a category $\mathcal{A}$, a $\mathcal{C}$-precover (respectively, a $\mathcal{C}$-cover with uniqueness) of an object X of $\mathcal{A}$ is a morphism $f: C \to X$, with C in $\mathcal{C}$, such that every morphism $g: C' \to X$ with C' in $\mathcal{C}$ can be factored (resp., in a unique way) through f. The dual concepts are those of $\mathcal{C}$-preenvelope and $\mathcal{C}$-envelope with uniqueness.The following is essentially known.

Proposition 2.1. *Let $\mathcal{A}$ be a locally finitely presented additive category, and let R be its associated functor ring. The following conditions are equivalent.*

(i) $\mathcal{A}$ *has products.*
(ii) R *is left locally coherent.*
(iii) $\mathcal{A}$ *has pseudocokernels.*
(iv) *Every right* R*-module has a flat preenvelope.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) are well known, as it is (iii) $\Rightarrow$ (ii). As for (ii) $\Rightarrow$ (iv), this was shown to hold for unital rings R in [8, Proposition 5.1]. It is not hard to see that the same arguments do work for the case of a general ring. □

As in the above result, an l.f.p. additive category $\mathcal{A}$ has pseudokernels if and only if every right R-module has a flat precover, R being its associated functor ring. In view of the theorem of Bican and Enochs [3], this always holds for rings with identity. We can then easily derive the same result for general rings (i.e., rings with enough idempotents).

Lemma 2.2. *Given a ring* R*, every unitary right* R*-module has a flat precover. Consequently, every locally finitely presented additive category has pseudokernels.*

Proof. Let D be the unital extension of the given ring R. Given a unitary right R-module M_R, it can be considered as a (unitary) right D-module. As such, it has a flat precover $F \to M$. Now, F is a right R-module and FR is a unitary flat right R-module which is easily seen to be a flat precover of M in the category $\mathrm{Mod}(R)$. □

We turn now to the problem of when an l.f.p. additive category $\mathcal{A}$ is abelian. Since $\mathcal{A}$ is always equivalent to $Fl(R)$, R being its associated functor ring, this problem is the same as that of studying when the category $Fl(R)$ of flat right R-modules is abelian. This has been answered, by means of properties of the category of all finitely presented objects $fp(\mathcal{A})$, by Crawley-Boevey [5], who refers the answer to Breitsprecher. Another answer was given in [12]: rings R for which $Fl(R)$ is abelian are called *right panoramic rings*, and they are characterized therein. We may add the following to the above mentioned characterizations. Recall that a right R-module M_R is said to be a torsion module in case $\mathrm{Hom}(M, F) = 0$ for every flat right R-module F.

Theorem 2.3. *Let* $\mathcal{A}$ *be a locally finitely presented additive category, and let* R *be its associated functor ring. The following conditions are equivalent.*
(1) $\mathcal{A}$ *is an abelian category.*
(2) *The following assertions hold.*
  (i) R *is left locally coherent and* $wD(R) \leq 2$.

(ii) *If M_R is a torsion module, then* $\mathrm{Ext}^1(M, F) = 0$ *for every flat module* F_R.

(3) *The following assertions hold.*

(i) *R is left locally coherent and $wD(R) \leq 2$.*

(ii) *If $f: L \to M$ is a monomorphism of right R-modules, and $g: F \to F'$ is the extension of f to the corresponding flat envelopes* (*with uniqueness*), *then g is a monomorphism.*

Proof. (1) $\Rightarrow$ (2) It follows from the above results that R is left locally coherent. Moreover, the fact that the category $\mathcal{A}$ has kernels and cokernels implies that the weak global dimension $wD(R) \leq 2$; in turn, this entails that every right R-module has a flat envelope with uniqueness.

Now, let M_R be a torsion module and take a projective presentation,

$$0 \to K \to P \to M \to 0$$

We have to show that each morphism $K \to F$, F being a flat module, may be extended to P. This will be proven if we see that the embedding $u: K \to P$ is a flat (pre)envelope. Take a projective module P' with an epimorphism $p: P' \to K$, which gives the composition $g = u \circ p$. We observe that this composition is an epimorphism of the category $Fl(R)$. Indeed, if there exists some morphism $h: P \to F'$ in $Fl(R)$ such that $h \circ g = 0$, then we infer that $h \circ u = 0$, from which it follows that h may be factored through the epimorphism $P \to M$. But M is torsion, whence $h = 0$. Next, since g is an epimorphism in $Fl(R)$, it is the cokernel of some morphism $f: F_0 \to P'$. It is not hard to derive from this that $K \to P$ is indeed a flat preenvelope.

(2) $\Rightarrow$ (3) Note that the class of torsion right R-modules is always closed under quotients, extensions and direct sums. Under the hypotheses (2), it is also closed under submodules. Therefore it is a hereditary torsion class, from which it follows that the injective envelope of any flat right R-module is again flat.

Next, we see that if M_R is a right R-module and $f: M \to F$ is its flat envelope (with uniqueness), then the cokernel and the kernel of f are torsion modules. First, if $g: F \to F'$ were a nonzero morphism between flat modules such that $g \circ f = 0$, then $0: M \to F'$ would have two different factorizations through F. Therefore the cokernel of f is torsion. Then, let K be the kernel of f, and let $M \xrightarrow{g} M' \xrightarrow{h} F$ be the epi-mono factorization of f. Since the cokernel of h is torsion, h is also a flat envelope. This implies that the canonical map $\mathrm{Hom}_R(M', F') \to \mathrm{Hom}_R(M, F')$ is an isomorphism for every flat right R-module F'. From this it follows that the canonical morphism $\mathrm{Hom}_R(M, F') \to \mathrm{Hom}_R(K, F')$ is zero, whenever F' is flat. A nonzero morphism $K \to F'$ would give a nonzero morphism $K \to E(F')$, that would extend to a morphism from M to $E(F')$. This contradiction proves that K is torsion. Hence the kernel and the cokernel of f are torsion modules.

Now, let $f: L \to M$ be a monomorphism of right R-modules, with flat envelopes $\alpha: L \to F$ and $\beta: M \to F'$. There is a unique morphism $g: F \to F'$ such that

$g \circ \alpha = \beta \circ f$. Let $K = \mathrm{Ker}(\beta)$, and let $K' = K \cap L$. Since K' is a torsion module, $\alpha(K') = 0$, from which it follows that K' is the kernel of α.

Let $X \subseteq F$ be the image of α and let $U = \mathrm{Ker}(g) \subseteq F$. Since K' is the kernel of $g \circ \alpha$, we have that $X \cap U = 0$. As the cokernel of α is a torsion module, so is the cokernel of the canonical inclusion $u\colon X \oplus U \to F$. Hence u is a flat envelope, and the inclusion $X \hookrightarrow F$ is a flat envelope as well. Now, let F'' be a flat module and $h\colon U \to F''$ a morphism. Then $0 \oplus h\colon X \oplus U \to F''$ has a unique factorization through u, $0 \oplus h = \delta \circ u$. Thus $\delta(X) = 0$. Since the inclusion $X \hookrightarrow F$ is a flat envelope, it follows that $\delta = 0$, and hence $h = 0$. This implies that U is a flat torsion module and thus g is a monomorphism.

(3) $\Rightarrow$ (1) As we have already remarked, the fact that R is left locally coherent and $wD(R) \leq 2$ implies that every module has a flat envelope with uniqueness. This gives a functor $\mathbf{a}\colon \mathrm{Mod}(R) \to Fl(R)$ which is clearly a left adjoint of the inclusion functor. The second hypothesis of (3) means that this functor preserves monomorphisms, and thus it is exact. By [19, Chapter X], we have that $Fl(R)$ is a Grothendieck category. $\square$

The question of when is the category $fp(\mathcal{A})$ an abelian category has also been studied by different authors. In particular, Gabriel [9, Proposition II.5] showed essentially, though with a different terminology, that if $fp(\mathcal{A})$ is abelian, then $\mathcal{A}$ is also abelian. The converse does not hold: it may happen that $\mathcal{A}$ is abelian, but $fp(\mathcal{A})$ is not, because the associated functor ring R fails to be right locally coherent. In fact, one has the following characterization of locally finitely presented additive categories $\mathcal{A}$ such that $fp(\mathcal{A})$ is abelian.

Proposition 2.4. *Let $\mathcal{A}$ be a locally finitely presented additive category with associated functor ring R. Then $fp(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is a locally coherent Grothendieck category if and only if R is two-sided panoramic.*

Proof. If $fp(\mathcal{A})$ is abelian, then $\mathcal{A}$ is abelian. Also, the associated functor ring R is right locally coherent. Since $\mathcal{A}$ is equivalent to $Fl(R)$, and this is a quotient category of $\mathrm{Mod}(R)$, it follows that $\mathcal{A}$ is locally coherent.

Conversely, let $\mathcal{A}$ be abelian and locally coherent. Let R be the associated functor ring, so that we may assume that $\mathcal{A} = Fl(R)$ and $fp(\mathcal{A}) = \mathrm{proj}(R)$. We are to show that R is right locally coherent; by the previous comments, we will have that $fp(\mathcal{A})$ is abelian.

To this end, let $P \to P'$ be a morphism in $\mathrm{proj}(R)$. If K is the kernel in $\mathrm{Mod}(R)$ of this morphism, it will suffice to show that K is finitely generated. But, since $wD(R) \leq 2$, we know that K is flat. As such, it is also the kernel of $P \to P'$ in the category $\mathcal{A}$. Therefore K is finitely generated in this category, as it is locally coherent. In this same category, K is a finitely generated subobject of the finitely presented object P. Again by the locally coherence of $\mathcal{A}$, we infer that K is finitely presented in $\mathcal{A}$, hence K is projective as a right R-module. In particular, it is finitely generated and we are done.

The equivalence of the condition that $\mathcal{A}$ is a locally coherent Grothendieck category and the fact that the associated functor ring R is right and left panoramic is a consequence of [12, Corollary 2.10]. □

If $\mathcal{A}$ is an l.f.p. additive category with associated functor ring R, then we know (see, for instance, [12, Corollary 3.5]) that the category $fp(\mathcal{A})$ of all locally finitely presented objects of $\mathcal{A}$ is equivalent to proj(R), the category of finitely generated projective right R-modules. This fact has the consequence that, in a sense, finitely presented objects of a locally finitely presented additive category $\mathcal{A}$ can be considered as finitely generated projective modules. We now state a simple application of this remark. First, we give a variation of the notion of almost split morphism.

DEFINITION 2.5. Let $\mathcal{A}$ be a locally finitely presented additive category, and let $f: M \to N$ be a morphism of $\mathcal{A}$. We say that f is a right fp-almost split morphism if f is not a split epimorphism, and for every $g: L \to N$ such that L is finitely presented and g is not a split epimorphism, one has that g may be factored through f.

Proposition 2.6. *Let N be a finitely presented object of the locally finitely presented additive category $\mathcal{A}$. The endomorphism ring of N is local if and only if there exists a morphism $f: M \to N$ of the category $\mathcal{A}$ that is a right fp-almost split morphism.*

Proof. Since N is an object of $fp(\mathcal{A})$, there is a corresponding finitely generated projective right R-module P through the equivalence between $fp(\mathcal{A})$ and proj(R), R being the functor ring associated to $\mathcal{A}$. Because of this equivalence, the endomorphism ring of N is local if and only if the endomorphism ring of P is local. We know from [1, Proposition 3.7] that this happens if and only if P has a unique maximal submodule. This, in turn, is equivalent to the existence of a morphism $g: F \to P$ with F a flat module, so that g is not an epimorphism, but every non-epimorphism $P' \to P$ of the category proj(R) factors through g. By using the equivalence between $\mathcal{A}$ and $Fl(R)$, we see that this property means precisely that there exists $f: M \to N$ which is not a retraction, and such that each morphism $L \to N$ that is not a retraction, factors through f. This proves the result. □

Given any locally finitely presented additive category $\mathcal{A}$, with associated functor ring R, the *pseudodual category* $p(\mathcal{A})$ of $\mathcal{A}$ is the only (up to equivalence) locally finitely presented additive category whose associated functor ring is R^{op} (see [7]). The duality between proj(R) and proj(R^{op}) entails that there is a duality between $fp(\mathcal{A})$

and $fp(p(\mathcal{A}))$.

In order to state the next result, we recall that an object X of an l.f.p. additive category $\mathcal{A}$ is finitely generated if there is an epimorphism $L \to X$ of the category $\mathcal{A}$, with L finitely presented. Then, we say that an object M of $\mathcal{A}$ is fp-injective if, given any pair of morphisms $f: L \to N$ and $g: L \to M$, where L is finitely generated, N is finitely presented, and f is a monomorphism, there exists $h: N \to M$ such that $g = h \circ f$.

Note that, in particular, if $\mathcal{A} = \mathrm{Mod}(R)$ for a given ring R (with enough idempotents), then a right R-module M is fp-injective in this sense if and only if it is fp-injective in the usual sense (see [20]).

Proposition 2.7. *Let $\mathcal{A}$ be a locally finitely presented additive category with associated functor ring R. The following conditions are equivalent.*
(1) *The pseudodual category $p(\mathcal{A})$ is equivalent to the category of right modules over a ring.*
(2) *R is left panoramic and every finitely presented object of $\mathcal{A}$ embeds in a finitely presented fp-injective object.*

Proof. Condition (2) means that the category $Fl(R^{op}) = p(\mathcal{A})$ is a Grothendieck category and $fp(p(\mathcal{A}))$ has a set of projective generators, in view of the duality between $fp(\mathcal{A})$ and $fp(p(\mathcal{A}))$. This amounts to the fact that $p(\mathcal{A})$ is a module category. □

Corollary 2.8. *Let $\mathcal{A}$ be a locally coherent category and let R be its associated functor ring. If R is left self-fp-injective, then $\mathcal{A}$ is equivalent to a module category.*

Proof. By [7, Corollary 2.6], the category $\mathcal{B} = p(\mathcal{A})$ is locally coherent. Now, the hypothesis implies that the functor ring R of this category $\mathcal{B}$ is right self-fp-injective. But it also left panoramic, and hence its pseudodual category is a module category by Proposition 2.7. Therefore $\mathcal{A}$ is equivalent to a module category. □

When $\mathcal{A} = \mathrm{Mod}(A)$ is a module category, then Proposition 2.7 has the following consequence.

Corollary 2.9. *Given a ring A, we have that the category $fp(\mathrm{Mod}(A))$ is dual to the category $fp(\mathrm{Mod}(B))$ for some ring B if and only if A is right locally coherent and every finitely presented right A-module embeds in a finitely presented fp-injective module.*

Proof. If $\mathrm{Mod}(A)$ is pseudodual to a module category, then its functor ring is left panoramic, by Proposition 2.7, hence it is two-sided panoramic. This implies that A is right locally coherent, by Proposition 2.4. The rest follows immediately from Proposition 2.7. □

3. Some classes of functor rings

We want next to characterize certain classes of rings through properties of the corresponding locally finitely presented additive categories. Frequently, properties of the functor ring R associated to the category $\mathcal{A}$ are given in terms of properties of the category $fp(\mathcal{A})$. For several classes of rings (self-fp-injective, coherent, semihereditary or IF rings), this has been done for unital rings R in [11] under the form of characterizing properties of the ring R through properties of the category $\mathrm{proj}(R)$, which is equivalent to $fp(\mathcal{A})$ (see, for example, [11, Propositions 1.3, 1.10, 1.12, Corollary 1.6]). We now add new classes of rings and new results to this study.

3.1. Semihereditary and perfect functor rings. The properties that a functor ring is hereditary or semihereditary are related to the existence of split kernels or cokernels in the corresponding additive category, as it was shown in [11, Propositions 1.9, 1.10]. We now add a property that characterizes left semihereditary functor rings R in terms of the right R-modules.

Proposition 3.1. *Let $\mathcal{A}$ be a locally finitely presented additive category, and let R be its associated functor ring. The following conditions are equivalent.*
(i) *R is left semihereditary.*
(ii) *The category $\mathcal{A}$ has cokernels, and every morphism in this category is the composition of a pure epimorphism followed by a monomorphism.*
(iii) *Each finitely presented right R-module M is the direct sum of a projective module and a torsion module.*

Proof. (i) $\Rightarrow$ (ii) By (i), R is left locally coherent and $wD(R) \leq 1$. Therefore every right R-module has a flat envelope with uniqueness (analogously to [2]). It follows that if $f\colon F \to F'$ is a morphism in $Fl(R)$, and $F' \to N$ is its cokernel in the category $\mathrm{Mod}(R)$, then a flat envelope of N provides the cokernel of f in the category $Fl(R)$. Since $Fl(R)$ is equivalent to the category $\mathcal{A}$, we see that $\mathcal{A}$ has cokernels. Moreover, the fact that $wD(R) \leq 1$ implies easily that the factorization $F \to M \to F'$ of f into an epimorphism followed by a monomorphism is such that M is flat. Thus $F \to M$ is a pure epimorphism of $Fl(R)$, which shows (ii).

(ii) $\Rightarrow$ (iii) Since every morphism in $Fl(R)$ has a cokernel, each right R-module has a flat envelope. The fact that every morphism of $Fl(R)$ is a pure epimorphism followed by a monomorphism implies that every flat envelope is an epimorphism. Now, the flat envelope of a finitely presented right R-module M is always a finitely generated projective module, from which it follows that M is isomorphic to the direct sum of a projective module and the kernel of its projective envelope, which is a torsion module.

(iii) $\Rightarrow$ (i) Let M be a finitely presented right R-module, so that $M = P \oplus T$, with P projective and T torsion. It follows that the projection $M \to P$ is a flat envelope and this shows that each finitely presented right R-module has a flat envelope which is an epimorphism. By using the fact that direct limits of epimorphisms are epimorphisms, we get that every right R-module has a flat envelope with uniqueness that is an epimorphism. Therefore every submodule of a flat right R-module is flat and it follows that R is left semihereditary. □

Right perfect functor rings correspond, as it is well known, to pure semisimple categories. A characterization of such categories, which extends a result known for unital rings [16], is the following.

Theorem 3.2. *A locally finitely presented category $\mathcal{A}$ with products is pure semisimple if and only if every object of $\mathcal{A}$ is isomorphic to a direct sum of indecomposable objects.*

Proof. The necessity of the condition follows from [20, 49.9, 49.10]. To prove the sufficiency, we assume that every object of $\mathcal{A}$ is a direct sum of indecomposables. If R is the functor ring associated to $\mathcal{A}$, then R is left locally coherent and hence the pseudodual category of $\mathrm{Mod}(R^{op})$, $\mathcal{D}(\mathcal{A})$, is a Grothendieck locally coherent category, according to [7, Corollary 2.6]. It follows from [5, Lemma 2] that there is a full and faithful functor $d\colon \mathcal{A} \to \mathcal{D}(\mathcal{A})$ whose image consists of all the fp-injective objects of $\mathcal{D}(\mathcal{A})$ so that any $d(U)$ is injective if and only if U is a pure injective object of $\mathcal{A}$. Moreover, d preserves direct sums and indecomposable objects. Thus, if E is any injective object of $\mathcal{D}(\mathcal{A})$, we have $E \cong d(M) \cong d(\bigoplus M_i) \cong \bigoplus d(M_i)$, where each $d(M_i)$ is indecomposable. This shows that every injective object of $\mathcal{D}(\mathcal{A})$ is a direct sum of indecomposable objects.

Next, the injective objects of $\mathcal{D}(\mathcal{A})$ form a set (when we take them up to isomorphism), because any indecomposable injective is the injective hull of some finitely generated object. Therefore there is a cardinal m so that every injective object of $\mathcal{D}(\mathcal{A})$ is m-generated. It follows from [10, Theorem 1.12] that each injective object of $\mathcal{D}(\mathcal{A})$ is Σ-injective. Now, let X be an arbitrary object of the category $\mathcal{A}$. Then $d(X)$ is fp-injective and hence it is a pure subobject of its injective hull E in $\mathcal{D}(\mathcal{A})$, by [10, Lemma 1.2]. Since E is Σ-injective we deduce by [10, Corollary 1.4] that $d(X)$ is injective. But then X is a pure injective object of $\mathcal{A}$, and thus $\mathcal{A}$ is a pure semisimple category. □

The following is an immediate consequence.

Corollary 3.3. *A left locally coherent ring R is right perfect if and only if every flat right R-module is a direct sum of indecomposable modules.*

3.2. IF functor rings. A ring R is called right IF when every injective right R-module is flat [4]. This is equivalent to the fact that every finitely presented right R-module embeds in a projective module. It was (essentially) shown in [11, Corollary 1.13] that R is right IF if and only if R is the functor ring of a locally finitely presented additive category $\mathcal{A}$ such that every morphism of the category $fp(\mathcal{A})$ is a pseudokernel. It is of interest to see what happens if we extend the condition that every morphism be a pseudokernel, from the category $fp(\mathcal{A})$ to the full category $\mathcal{A}$.

Proposition 3.4. *Let $\mathcal{A}$ be a locally finitely presented additive category, and let R be its associated functor ring. Then every morphism of $\mathcal{A}$ is a pseudokernel if and only if R is right perfect and right IF.*

Proof. Necessity. Let F_R be flat, and let $f\colon P \to F$ be any epimorphism of $\mathrm{Mod}(R)$ with P projective. Since f has to be a pseudokernel, it is clear that f is a pseudokernel of 0. But then $1\colon F \to F$ can be factored through f, from which it follows that R is right perfect. Then, any morphism $\alpha\colon P_1 \to P_0$ of the category $fp(\mathcal{A}) = \mathrm{proj}(R)$ is a pseudokernel in $\mathcal{A} = Fl(R)$; say α is a pseudokernel of $\beta\colon P_0 \to P$. If $K = \mathrm{Ker}(\beta)$, it follows that α can be factored in $P_1 \to K \to P_0$, the first morphism being epimorphism. Thus $0 \to K \to P_0 \to \mathrm{Im}(\beta) \to 0$ is short exact in $\mathrm{Mod}(R)$. Since $\mathrm{Im}(\beta)$ embeds in a projective module P, it embeds in a finitely presented projective P'. We then infer that α is also a pseudokernel of $P_0 \to P'$ in the category $\mathrm{proj}(R)$. This shows that R is right IF.

Sufficiency. Let $f\colon P_1 \to P_0$ be any morphism of the category $Fl(R)$. By hypothesis, its cokernel C in $\mathrm{Mod}(R)$ can be embedded in a projective module P. Then, the composite morphism $P_0 \to P$ has f as a pseudokernel. □

This class of rings has appeared before in Harada [14] as the perfect rings with property (II). These rings have very good properties in connection to duality. Namely, we have that this class is closed under pseudoduality.

Theorem 3.5. *Let R be a right perfect and right IF ring. Then, the pseudodual category of* $\mathrm{Mod}(R)$ *is equivalent to a module category* $\mathrm{Mod}(S^{op})$*, and S is again a right perfect right IF ring.*

Proof. Note that if R is right perfect and right IF, then every injective right R-module is projective, and R is locally right noetherian, by [14]. Moreover, every finitely presented right R-module M embeds in some injective module, which is therefore a finite direct sum of indecomposable finitely generated projective modules. As a consequence, each finitely presented right R-module embeds in a finitely presented fp-injective module, so that Mod(R) is pseudodual to a module category Mod(S^{op}) which is locally coherent by Corollary 2.9.

Take now any finitely presented left S-module ${}_SN$. Since the category fp(Mod(S^{op})) is dual to the category fp(Mod(R)), and R is right locally noetherian, we see that ${}_SN$ has the descending chain condition on finitely presented subobjects. Since S is left locally coherent, ${}_SN$ has the descending chain condition on finitely generated submodules. By applying [20], we infer that S is right perfect.

Next, let ${}_SP$ be a finitely generated projective left S-module. Since ${}_SP$ is a projective object of the category fp(Mod(S^{op})), one gets by the duality, that the corresponding object Q_R is injective in the category fp(Mod(R)), so that Q_R is a finitely presented fp-injective module. Consequently, Q_R is injective and projective, because R is right locally noetherian, right IF and right perfect. This implies that ${}_SP$ is injective in fp(Mod(S^{op})) and hence ${}_SP$ is fp-injective. We have thus shown that S is left self-fp-injective, from which it follows that it is right IF. □

We say that a ring S is the *left pseudodual* of the ring R when the categories Mod(S^{op}) and Mod(R) are pseudodual categories.

4. Self-pseudodual rings

It is well known that for a unital QF ring R, the categories of finitely presented right and left modules are dual to each other. We shall say that a ring R is self-pseudodual in case R is the left pseudodual of itself. Therefore QF unital rings are self-pseudodual. In [14], Harada defined (right) QF rings as being rings R such that every projective right R-module is injective, and he studied the relationship between these rings and those having every injective module projective, i.e., rings with property (II). On the other hand, Garkusha and Generalov [13] studied unital rings R which are self-pseudodual with the duality between finitely presented right and left R-modules being given by the $\mathrm{Hom}_R(-, R)$ functors. We may use functor rings and the theory of pseudoduality considered above to study self-pseudodual rings.

Theorem 4.1. *Let R be any ring. R is self-pseudodual if and only if the categories $Fl(R)$ of flat right R-modules and $Fpi(R)$ of fp-injective right R-modules are equivalent categories.*

Proof. Let R be self-pseudodual. By [7], R is right locally coherent. If $\mathcal{A} = Fl(R)$, then R is the functor ring of $\mathcal{A}$ and $\mathcal{A}$ is a category with products. By [17], $\mathcal{A}$ is also an exactly definable category and it consists of the fp-injective objects of $\mathcal{D}(\mathcal{A})$. By construction, $\mathcal{D}(\mathcal{A})$ is the pseudodual category to Mod(R^{op}), that is, $\mathcal{D}(\mathcal{A})$ is equivalent to Mod(R). This shows that $\mathcal{A} = Fl(R)$ and $Fpi(R)$ are equivalent categories.

Assume now that $Fl(R)$ and $Fpi(R)$ are equivalent categories. Since $Fpi(R)$ is a category with products, so is $Fl(R)$. This implies that both categories are exactly definable and locally finitely presented additive categories. Therefore R is the functor ring of $Fpi(R)$, while Mod(R) is equivalent also to $\mathcal{D}(Fpi(R))$. But this category is pseudodual to the category Mod(R^{op}), which means that R is indeed self-pseudodual. □

This suggests the following generalization of QF rings. We shall say that a ring R is a (right) PIE-ring (for "projectives and injectives are equivalent") if the categories Proj(R) and Inj(R) of projective and injective right R-modules are equivalent categories.

Theorem 4.2. *A ring R is a right PIE-ring if and only if R is right perfect, right locally noetherian and self-pseudodual.*

Proof. If R satisfies the conditions, then it is clear that Proj(R) = $Fl(R)$, Inj(R) = $Fpi(R)$ and both categories are equivalent by Theorem 4.1.

Conversely, suppose that R is right PIE. We have to show that it is right perfect and right locally noetherian. By hypothesis, Inj(R) is a category with direct sums. Now, let $\{E_i\}_{i\in I}$ be any family of injective right R-modules, $M = \bigoplus_I E_i$ and let E be the injective envelope of M. On the other hand, there is an injective module U which is the direct sum of the E_i in the category Inj(R). An standard argument shows that, under these hypotheses, there is a canonical isomorphism $E \cong U$. Suppose that $E/M \neq 0$, and let F be the injective envelope of E/M. There is an induced nonzero morphism $U \to F$, such that, for every index $i \in I$, the composition $E_i \to U \to F$ gives zero. But this is a contradiction. Therefore $M = E$ and the class of injective right R-modules is closed under direct sums. This implies that R is right locally noetherian.

Then, Proj(R) is a category with products. Let $\{P_i\}_{i\in I}$ be any family of projective right R-modules, P is the product of this family in the category Proj(R), and M is its product in Mod(R). There is a unique morphism $\alpha\colon P \to M$ commuting with the canonical projections. On the other hand, for any epimorphism $\pi\colon Q \to M$ with Q projective, there is a unique $\beta\colon Q \to P$ commuting with the defined morphisms. Let $L = \operatorname{Ker}\pi$, and pretend that $\beta(L) \neq 0$. Then, there exists a projective module Q' and a morphism $\gamma\colon Q' \to Q$ so that $\beta\gamma \neq 0$. Moreover, $\beta\gamma$ composed with the canonical projections $P \to P_i$ gives zero. The uniqueness of the morphism in the definition of the product implies that $\beta\gamma = 0$ and we get a contradiction. Therefore $\beta(L) = 0$ and we get a factorization of β through a morphism $\delta\colon M \to P$. Using now δ and α we see that they are inverse isomorphisms, so that M is projective. Consequently, the product of projective right R-modules is projective and R is right perfect and left locally coherent.

Finally, R is self-pseudodual by Theorem 4.1. □

In fact, a self-pseudodual ring R that is either right perfect or right locally noetherian is right PIE. Because, by the pseudoduality, R is right and left locally coherent and the descending chain condition on finitely generated submodules of each finitely presented left R-module is equivalent to the ascending chain condition on finitely generated submodules of each finitely presented right R-module.

Proposition 4.3. *Let R be a right PIE ring. The following conditions are equivalent.*
(1) *R is right locally finite.*
(2) *R is left PIE.*
(3) *The pseudoduality between* $\mathrm{Mod}(R)$ *and* $\mathrm{Mod}(R^{op})$ *is a Morita duality.*

Proof. (1) $\Rightarrow$ (2) If R is right locally finite, then it is left perfect. The pseudoduality implies that R is also left locally noetherian, which means that it is left PIE.

(2) $\Rightarrow$ (1) R being left perfect and right locally noetherian, we have immediately that R is right locally artinian.

(2) $\Rightarrow$ (3) The categories $fp(\mathrm{Mod}(R))$ and $fp(\mathrm{Mod}(R^{op}))$ contain precisely all the finitely generated submodules of R and R^{op}, respectively, and they are closed under submodules and quotients. Therefore it is a Morita duality.

(3) $\Rightarrow$ (2) If X is a submodule of a finitely presented left R-module M, we have an epimorphism in the dual category which shows that X corresponds to a finitely generated, hence finitely presented, right R-module. Since the dual of a finitely presented is finitely presented, we infer that X is finitely presented. Therefore R is left locally noetherian and R is left PIE. $\square$

We finish by stating a problem connected with these results.

Question. Is any right PIE-ring a left PIE-ring?

This question is connected to the following idea, introduced in [7]. Given a locally finitely presented additive category $\mathcal{A}$ with associated functor ring R, we call the symmetric category $s(\mathcal{A})$ of $\mathcal{A}$ to a locally finitely presented category (which, if it exists, is uniquely determined up to equivalence) whose associated functor ring S is such that the categories $\mathrm{Mod}(R^{op})$ and $\mathrm{Mod}(S^{op})$ are pseudodual categories. In view of the Gruson-Jensen duality and its generalization (see [7, Theorem 2.9]), if $\mathcal{A}$ is the module category $\mathrm{Mod}(A)$, where A is a ring with enough idempotents, then $s(\mathcal{A})$ is $\mathrm{Mod}(A^{op})$.

The connection is the following. The functor ring R is self-pseudodual precisely if the category $s(\mathcal{A})$ exists and its functor ring is R^{op}, hence if $s(\mathcal{A})$ is equivalent to the pseudodual category $p(\mathcal{A})$. Therefore R is a right PIE-ring if and only if it is the functor ring of a pure semisimple category $\mathcal{A}$ such that $\mathcal{A}$ and $s(\mathcal{A})$ are pseudodual categories. So, our question is equivalent to the following.

Let $\mathcal{A}$ be a pure semisimple category such that $\mathcal{A}$ and its symmetric category $s(\mathcal{A})$ are pseudodual categories. Is $s(\mathcal{A})$ also a pure semisimple category? We know that Herzog's theorem [15] shows that the answer is affirmative when $\mathcal{A}$ is the category of right modules over a unital ring. By [7], the answer is still yes if $\mathcal{A}$ is the category of right modules over a ring A and the pseudoduality is given through the functor $\mathrm{Hom}_A(-, A)$.

ACKNOWLEDGEMENT. The first and third authors acknowledge support from the DGES of Spain.

References

[1] P.N. Anh, N.V. Loi and D. Thanh: *Perfect rings without identity*, Tsukuba J. Math. **11** (1987), 1–16.
[2] J. Asensio Mayor and J. Martínez Hernández: *On flat and projective envelopes*, J. Algebra **160** (1993), 434–440.
[3] L. Bican, R. El Bashir and E. Enochs: *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), 385–390.
[4] R.R. Colby: *Rings which have flat injective modules*, J. Algebra **35** (1975), 239–252.
[5] W. Crawley-Boevey: *Locally finitely presented additive categories*, Comm. Algebra **22** (1994), 1641–1674.
[6] N. Ding: *On envelopes with the unique mapping property*, Comm. Algebra **24** (1996), 1459–1470.
[7] N.V. Dung and J.L. García: *Additive categories of locally finite representation type*, J. Algebra **238** (2001), 200–238.
[8] E.E. Enochs: *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 189–209.
[9] P. Gabriel: *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.
[10] J.L. García and N.V. Dung: *Some decomposition properties of injective and pure-injective modules*, Osaka J. Math. **31** (1994), 95–108.
[11] J.L. García, J. Martínez Hernández and P.L. Gómez Sánchez: *Endomorphism rings and the category of finitely generated projective modules*, Comm. Algebra **28** (2000), 3837–3852.
[12] J.L. García and D. Simson: *On rings whose flat modules form a Grothendieck category*, Colloq. Math. **73** (1997), 115–141.
[13] G.A. Garkusha and A.I. Generalov: *Duality for categories of finitely presented modules*, St. Petersburg Math. J. **11** (2000), 1051–1061.
[14] M. Harada: *Perfect categories* IV, Osaka J. Math. **10** (1973), 585–596.
[15] I. Herzog: *A test for finite representation type*, J. Pure Appl. Algebra **95** (1994), 151–182.
[16] B. Huisgen-Zimmermann: *Rings whose right modules are direct sums of indecomposable modules*, Proc. Amer. Math. Soc. **77** (1979), 191–197.
[17] H. Krause: *Exactly definable categories*, J. Algebra **201** (1998), 456–492.
[18] D. Simson: *On pure semisimple Grothendieck categories* I, Fund. Mat. **100** (1978), 211–222.
[19] B. Stenstr$\bar{\text{o}}$m: Rings of quotients, Springer, 1975.
[20] R. Wisbauer: Foundations of module and ring theory, Gordon and Breach, 1991.

J.L. García
Department of Mathematics, University of Murcia
300071 Murcia, Spain
e-mail: jlgarcia@um.es

J. Martínez Hernández
Department of Mathematics, University of Murcia
300071 Murcia, Spain
e-mail: juan@um.es

P.L. Gómez Sánchez
Departamento de Matemática Aplicada
Universidad Politécnica de Cartagena
Cartagena, Spain
e-mail: pedroluis.gomez@upct.es